المحاضرة الثانية عملي :

إذا كان $1 \leq P > 0$ وكان $a_1, a_2, \ldots, a_n \geq 0$ و $b_1, b_2, \ldots, b_n \geq 0$

عندئذ :

أثبت أنه :
$$\sum_{k=1}^{n} (a_k+b_k)^P \leq \sum_{k=1}^{n} a_k^P + \sum_{k=1}^{n} b_k^P$$

الحل : سنثبت في البداية :
$$(a+b)^P \leq a^P + b^P$$

سنستقدم التابع :
$$f(t) = 1+t^P-(1+t)^P \qquad t \geq 0$$
$$f'(t) = P t^{P-1} - P(1+t)^{P-1}$$
$$f'(t) = \{t^{P-1}-(1+t)^{P-1}\} \geq 0$$
$$f(t) \geq f(0)$$
$$f(t) \geq 0$$
$$1+t^P \geq (1+t)^P$$
$$\left(1+\frac{a}{b}\right)^P \leq 1+\frac{a^P}{b^P} \qquad t = \frac{a}{b}$$
$$\frac{(a+b)^P}{b^P} \leq \frac{a^P+b^P}{b^P} \Rightarrow (a+b)^P \leq a^P + b^P$$

نستبدل كل a بـ a_k

وكل b بـ b_k
$$|a_k+b_k|^P \leq |a_k|^P + |b_k|^P$$
$$\sum_{k=1}^{n} |a_k+b_k|^P \leq \sum_{k=1}^{n} |a_k|^P + \sum_{k=1}^{n} |b_k|^P$$

أثبت أنه :
$$\left(\sum_{k=1}^{n} |a_k|\right)^P \leq N^{P-1} \sum_{k=1}^{n} |a_k|^P \qquad ; P \geq 1$$

الحل :
$$\left(\sum_{k=1}^{n} |a_k|\right)^P = \left(\sum_{k=1}^{n} |a_k \cdot 1|\right)^P \leq$$

KORANİ

الموضوع

$$\leq \left(\left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{n} (1)^q \right)^{\frac{1}{q}} \right)^p$$

$\frac{1}{p} + \frac{1}{q} = 1$

$$= N^{\frac{p}{q}} \sum_{k=1}^{n} |a_k|^p = N^{p-1} \sum_{k=1}^{n} |a_k|^p$$

$\frac{p+q}{pq} = 1 \Rightarrow p+q = pq$

$p = q(p-1)$

$\frac{p}{q} = p-1$

إذا كان محولة حسب ستيلجس:

$$\int_a^b |f(x)|^p \, d(\varphi(x)) = 1$$

$$\int_a^b |g(x)|^q \, d(\varphi(x)) = 1$$

أثبت أن:

$$\int_a^b |f(x) \cdot g(x)| \, d(\varphi(x)) \leq 1$$

الحل:

من المتراجحة

$$|A \cdot B| \leq \frac{|A|^p}{p} + \frac{|B|^q}{q}$$

$$A = f(x) \ , \ B = g(x)$$

$$|f(x) \cdot g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}$$

نكامل طرفي المتراجحة على المجال $[a,b]$ حسب ستيلجس بالنسبة لـ $\varphi(x)$

$$\int_a^b |f(x) \cdot g(x)| \, d(\varphi(x)) \leq \int_a^b \frac{|f(x)|^p}{p} \, d(\varphi(x)) + \int_a^b \frac{|g(x)|^q}{q} \, d(\varphi(x))$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\int_a^b |f(x) \cdot g(x)| \, d(\varphi(x)) \leq \left(\underbrace{\int_a^b |f(x)|^p \, d(\varphi(x))}_{1} \right)^{\frac{1}{p}} \cdot \left(\underbrace{\int_a^b |g(x)|^q \, d(\varphi}_{1} \right.$$

4- إذا علمت أنه : $s>0,\ t>0$ حيث : $r\geq 1$ أعداد حقيقية موجبة [illegible]

$$\frac{1}{r}+\frac{1}{t}+\frac{1}{s}=1$$

$a_1,a_2,\dots,a_n\geq 0$

$b_1,b_2,\dots,b_n\geq 0$

$c_1,c_2,\dots,c_n\geq 0$

أثبت أنه :

$$\sum_{k=1}^{n}|a_k.b_k.c_k|\leq\left(\sum_{k=1}^{n}|a_k|^r\right)^{\frac{1}{r}}.\left(\sum_{k=1}^{n}|b_k|^s\right)^{\frac{1}{s}}.\left(\sum_{k=1}^{n}|c_k|^t\right)^{\frac{1}{t}}$$

الحل. سنثبت أنه :

$$|A.B.C|\leq\frac{|A|^r}{r}+\frac{|B|^s}{s}+\frac{|C|^t}{t}$$

$$|A.B.C|\leq\frac{|A|^r}{r}+\frac{|B.C|^q}{q}\ ;\ \frac{1}{r}+\frac{1}{q}=1$$

$$=\frac{|A|^r}{r}+\frac{|B|^q.|C|^q}{q}\leq$$

$A=|B|^q$

$B=|C|^q$

$$\leq\frac{|A|^r}{r}+\frac{1}{q}\left(\frac{|B|^{qs_1}}{s_1}+\frac{|C|^{qt_1}}{t_1}\right)\ ;\ \frac{1}{s_1}+\frac{1}{t_1}=1$$

$$|A.B.C|\leq\frac{|A|^r}{r}+\frac{|B|^s}{s}+\frac{|C|^t}{t}$$

$qs_1=s$

$qt_1=t$

$$\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=\frac{1}{r}+\frac{1}{qs_1}+\frac{1}{qt_1}=$$

$$=\frac{1}{r}+\frac{1}{q}\underbrace{\left(\frac{1}{s_1}+\frac{1}{t_1}\right)}_{1}=\underbrace{\frac{1}{r}+\frac{1}{q}}_{1}=1$$

نفرض :

$$A=\frac{|a_k|}{\left(\sum_{k=1}^{n}|a_k|^r\right)^{\frac{1}{r}}}\ ,\ B=\frac{|b_k|}{\left(\sum_{k=1}^{n}|b_k|^s\right)^{\frac{1}{s}}}\ ,\ C=\frac{|c_k|}{\left(\sum_{k=1}^{n}|c_k|^t\right)^{\frac{1}{t}}}$$

بالتعويض نجد :

$$\frac{|a_k.b_k.c_k|}{\left(\sum_{k=1}^{n}|a_k|^r\right)^{\frac{1}{r}}.\left(\sum_{k=1}^{n}|b_k|^s\right)^{\frac{1}{s}}.\left(\sum_{k=1}^{n}|c_k|^t\right)^{\frac{1}{t}}}\leq\frac{|a_k|^r}{r.\sum_{k=1}^{n}|a_k|^r}+\frac{|b_k|^s}{s\left(\sum_{k=1}^{n}|b_k|^s\right)}+\frac{|c_k|^t}{t\left(\sum_{k=1}^{n}|c_k|^t\right)}$$

الموضوع

باعتبار المجموع من 1 إلى N نحصل على المتراجحة المطلوبة.

ت- أثبت صحة المتراجحة التالية:

$$\sum_{k=1}^{n}|a_k| \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}$$

$1 \leq p < \infty$ و

الحل: نفرض أن المتراجحة السابقة صحيحة من أجل n

$$\sum_{k=1}^{n}|a_k| \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}$$

$$(a+b)^p \leq a^p + b^p$$

$$(a+b) \leq (a^p+b^p)^{\frac{1}{p}}$$

نثبت صحتها من أجل $n+1$:

$$\sum_{k=1}^{n+1}|a_k| = \sum_{k=1}^{n}|a_k| + |a_{n+1}| \leq \left(\left(\sum_{k=1}^{n}|a_k|\right)^p + |a_{n+1}|^p\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\sum_{k=1}^{n}|a_k|^p + |a_{n+1}|^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{n+1}|a_k|^p\right)^{\frac{1}{p}}$$

الفضاءات المترية الشهيرة:

المسافات في الفضائين: $\mathbb{C}^n, \mathbb{R}^n$

1- $$d_1(x,y) = \left(\sum_{i=1}^{n}|x_i - y_i|^2\right)^{\frac{1}{2}}$$

2- $$d_2(x,y) = \sum_{i=1}^{n}|x_i - y_i|$$

3- $$d_3(x,y) = \operatorname{Max}|x_i - y_i|$$

1- الشرط الرابع:

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$d(x,z) = \left(\sum_{i=1}^{n}|x_i - z_i|^2\right)^{\frac{1}{2}} =$$

$$= \left(\sum_{i=1}^{n}|x_i - y_i + y_i - z_i|^2\right)^{\frac{1}{2}}$$

$$\left(\sum_{k=1}^{n}|a_k+b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}|b_k|^p\right)^{\frac{1}{p}}$$

تطبيق متراجحة مينكوفسكي

4- فضاء l^p:
$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$$
نطبق متراجحة منكوفسكي لإثبات الشرط الرابع.

5- فضاء l^∞:
$$d(x,y) = \sup_{N} |x_n - y_n|$$
إذا لم تبلغ الدالة الحد تكون sup
وإذا بلغت الحد الأعلى تكون Max

6- الفضاء $C[a,b]$: نتعرف عليه مسافتين:
$$d_1(x,y) = \max_{a \le t \le b} |x(t) - y(t)|$$
$$d_2(x,y) = \int_a^b |x(t) - y(t)|\, dt$$

7- الفضاء $L^p[a,b]$
$$f(x) \in L^p[a,b]$$
$$\int_a^b |f(x)|^p\, dx < +\infty$$
$$d(f,g) = \left[\int_a^b |f(t) - g(t)|^p\, dt\right]^{\frac{1}{p}}$$ نطبق متراجحة منكوفسكي بالشرط الرابع.

8- الفضاء $L^\infty[a,b]$:
$$d(x,y) = \underset{x \in [a,b]}{\text{ess}\ \sup} |f(x) - g(x)|$$
$$= \inf\{c;\ \underset{\text{قياس ليبيغ}}{\lambda}\{x \in [a,b];\ |f(x)| \ge c\} = 0\}$$

9- الفضاء $S[a,b]$
$$d(f,g) = \int_a^b \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx$$

10- الفضاء $BV[a,b]$
$$d(f,g) = |f(a) - g(a)| + V_a^b(f-g)$$
$$f = g \Leftarrow d(f,g) = 0$$
$|f(a) - g(a)| = 0$, $V_a^b(f-g) = 0$ الدالة هنا هي الدالة الثابتة
$f(a) = g(a)$

الموضوع

6 إذا علمت أن:

$$d(x,y)=\left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{\frac{1}{2}}$$

$x=(x_1,x_2,\ldots,x_n)\in R^n$ $\quad e(x,y)=\sum_{i=1}^{n}|x_i-y_i|$

$y=(y_1,y_2,\ldots,y_n)\in R^n$ $\quad c(x,y)=\underset{1\le i\le n}{Max}|x_i-y_i|$

المطلوب: اثبت المتراجحات التالية:

1) $c(x,y)\le d(x,y)\le\sqrt{n}\,c(x,y)$

الحل:

$$c(x,y)=\underset{1\le i\le n}{Max}|x_i-y_i|=\left(\underset{1\le i\le n}{(Max}|x_i-y_i|^2)\right)^{\frac{1}{2}}=\left(\underset{1\le i\le n}{Max}|x_i-y_i|^2\right)^{\frac{1}{2}}\le$$

$$\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}}=d(x,y)$$

$$d(x,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}}\le\left(\sum_{i=1}^{n}(\underset{1\le i\le n}{Max}|x_i-y_i|^2)\right)^{\frac{1}{2}}=N^{\frac{1}{2}}\left(\underset{1\le i\le n}{Max}|x_i-y_i|\right)^{\frac{2}{2}}$$

$$=\sqrt{N}\,c(x,y)$$

2) $c(x,y)\le e(x,y)\le N\,c(x,y)$

$$c(x,y)=\underset{1\le i\le n}{Max}|x_i-y_i|\le\sum_{i=1}^{n}|x_i-y_i|=e(x,y)$$

$$e(x,y)=\sum_{i=1}^{n}|x_i-y_i|\le\sum_{i=1}^{n}\underset{1\le i\le n}{Max}|x_i-y_i|=N\,c(x,y)$$

3) $d(x,y)\le e(x,y)\le\sqrt{N}\,d(x,y)$

$$d(x,y)=\left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{\frac{1}{2}}$$

$n=1$ $\quad ((x_1-y_1)^2)^{\frac{1}{2}}\le|x_1-y_1|$

نفرض أنه صحيحة من أجل n ولنثبت صحتها من أجل n+1 :

$$\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n}|x_i-y_i|$$

$$\left(\sum_{i=1}^{n+1}|x_i-y_i|^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n}|x_i-y_i|^2+|x_{n+1}-y_{n+1}|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\left(\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}}\right)^2+\left(\left(|x_{n+1}-y_{n+1}|^2\right)^{\frac{1}{2}}\right)^2+\right.$$

$$\left.+2\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}}\cdot\left(|x_{n+1}-y_{n+1}|^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}}+|x_{n+1}-y_{n+1}| \leq$$

$$\sum_{i=1}^{n}|x_i-y_i|+|x_{n+1}-y_{n+1}| = \sum_{i=1}^{n+1}|x_i-y_i|$$